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Twist Deformation of the rank one Lie Superalgebra

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Abstract

The Drinfeld twist is applyed to deforme the rank one orthosymplectic Lie superalgebra $osp(1|2)$. The twist element is the same as for the $sl(2)$ Lie algebra due to the embedding of the $sl(2)$ into the superalgebra $osp(1|2)$. The R-matrix has the direct sum structure in the irreducible representations of $osp(1|2)$. The dual quantum group is defined using the FRT-formalism. It includes the Jordanian quantum group $SL_\xi(2)$ as subalgebra and Grassmann generators as well.

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1 The deformed algebra $osp_\xi(1|2)$

It is difficult to overestimate the role of the rank one Lie algebra $sl(2)$ in the theory of Lie groups and their applications. The corresponding role for Lie superalgebras is played by the orthosymplectic superalgebra $osp(1|2)$ with five generators $\{h, X_-, X_+, v_-, v_+\}$ and commutation relations (Lie super- or $\mathbf{Z}_2$ graded-brackets):

$$[h, X_\pm] = \pm 2X_\pm, \quad [X_+, X_-] = h \, , \tag{1}$$

$$[h, v_\pm] = \pm v_\pm, \quad [v_+, v_-]_+ = -h/4 \, , \tag{2}$$

$$[X_\pm, v_\pm] = 0, \quad [X_\pm, v_\mp] = v_\pm, \quad [v_\pm, v_\pm]_+ = \pm X_\pm/2 \, . \tag{3}$$

The generators h and $X_\pm$ are even (zero parity $p = 0$), while $v_\pm$ are odd, $p = 1$. As a Hopf superalgebra, the universal enveloping $\mathcal{U}(osp(1|2))$ of $osp(1|2)$ is generated, as $sl(2)$, just by three elements: it is sufficient to start from $\{h, v_-, v_+\}$ restricted by the relations (2) only, and define $X_\pm \equiv \pm 4v_\pm^2$.

The quantum deformation of $sl(2)$ can be considered as a "pivot" of the quantum group theory [1, 2], while the corresponding quantum superalgebra $osp_q(1|2)$ constructed in [3, 4, 5], is the corresponding analogue for the quantum supergroups. As a quasitriangular Hopf superalgebra $osp_q(1|2)$, analogously to the universal enveloping of $osp(1|2)$, is generated by three elements $\{h, v_-, v_+\}$ under the relations

$$[h, v_\pm] = \pm v_\pm, \qquad [v_+, v_-] = -\frac{1}{4}(q^h - q^{-h})/(q - q^{-1}).$$

It is worthy to note that, while $sl(2)$ is embedded into $osp(1|2)$, such embedding does not exist for $sl_q(2)$ into $osp_q(1|2)$ because the coproduct of even elements $X_\pm \sim v_\pm^2$ includes also odd ones.

The aim of this paper is to construct and study the twist deformation [6] of $osp(1|2)$ that looks, in some sense, more natural than $osp_q(1|2)$ because it is consistent with this fundamental property of inclusion $sl(2) \subset osp(1|2)$ and it is generated by the same twist element of $sl(2)$.

The triangular Hopf algebra $sl_\xi(2)$ (cf. [7, 8, 9, 10, 11, 12], and Refs therein) is given by the extension of the twist deformation of the universal

enveloping of the Borel sub-algebra $B_- \equiv \{h, X_-\}$ to the whole $\mathcal{U}(sl(2))$. The twist element $\mathcal{F}$ is

$$\mathcal{F} = 1 \; + \; \xi \, h \otimes X_- \;\; + \;\; \frac{\xi^2}{2} \, h(h+2) \otimes X_-^2 \; + \; \dots$$

that can be written as

$$\mathcal{F} \;=\; (1 - 2\,\xi \; 1 \otimes X_-)^{-\frac{1}{2}(h \otimes 1)} \;=\; \exp(\frac{1}{2} \; h \otimes \sigma) \tag{4}$$

where $\sigma = \; -ln\,(1 - 2\,\xi\, X_-)$.

Let us recall from [6] that for a quasitriangular Hopf algebra $\mathcal{A}$ with an R-matrix $\mathcal{R}$ the twisted Hopf algebra $\mathcal{A}_t$ has R-matrix $\mathcal{R}^{(\mathcal{F})}$ given by the twist transformation

$$\mathcal{R}^{(\mathcal{F})} = \mathcal{F}_{21} \mathcal{R} \mathcal{F}^{-1} \tag{5}$$

of the original R-matrix $\mathcal{R}$, where $\mathcal{F}_{21} = \mathcal{P}\mathcal{F}\mathcal{P}$, and $\mathcal{P}$ is the permutation map in $\mathcal{A} \otimes \mathcal{A}$. The algebraic sector of $\mathcal{A}_t$ is not changed, and new coproduct is $\Delta_t = \mathcal{F} \Delta \mathcal{F}^{-1}$. The twist element satisfies the relations in $\mathcal{A} \otimes \mathcal{A}$ [6]

$$(\epsilon \otimes id)\, \mathcal{F} = (id \otimes \epsilon)\, \mathcal{F} = 1 \,,$$

and in $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$

$$\mathcal{F}_{12}\, (\Delta \otimes id)\, \mathcal{F} = \mathcal{F}_{23}\, (id \otimes \Delta)\, \mathcal{F} \,.$$

According to this Drinfeld definition, the algebraic relations of eqs. (1) for the twisted $sl(2)$ are still the same, while the twisted coproduct $\Delta_t \equiv \mathcal{F} \Delta \mathcal{F}^{-1}$ is now on the generators

$$\begin{aligned}
\Delta_t(h) \;\; &= \;\; h \otimes e^{\sigma} + 1 \otimes h \;, \\
\Delta_t(X_-) \;\; &= \;\; X_- \otimes 1 + 1 \otimes X_- - 2\,\xi\; X_- \otimes X_- = \; X_- \otimes e^{-\sigma} + 1 \otimes X_- \,, \\
\Delta_t(X_+) \;\; &= \;\; X_+ \otimes e^{\sigma} + 1 \otimes X_+ - \xi\, h \otimes e^{\sigma} h + \frac{\xi}{2}\, h(h-2) \otimes e^{\sigma} (1 - e^{\sigma}).
\end{aligned}$$

Let us stress that this twist of the whole $sl(2)$ is obtained due to the embedding $B_- \subset sl(2)$.

Thus, knowing that $B_- \subset sl(2) \subset osp(1|2)$, the procedure can be simply iterated to find $osp_{\xi}(1|2)$ (as well as the twisted deformations of all others nontrivial embeddings of B_-). It is an easy exercise, keeping in mind the

expression of $\mathcal{F}$ (eq. (4)), commutation relations (2), (3) and the primitive coproduct of $osp(1|2)$, to obtain:

$$\begin{aligned} \Delta_t(h) &= h \otimes e^{\sigma} + 1 \otimes h \,, \\ \Delta_t(v_-) &= v_- \otimes e^{-\sigma/2} + 1 \otimes v_- \,, \\ \Delta_t(v_+) &= v_+ \otimes e^{\sigma/2} + 1 \otimes v_+ + \xi h \otimes v_- e^{\sigma} . \end{aligned} \tag{6}$$

One can reproduce the coproducts of $X_\pm$ by squaring the coproducts of $v_\pm$, taking into account the Z_2-grading of tensor product:

$$(x \otimes y)\,(u \otimes w) = (-1)^{p(u)p(y)}(x\, u \otimes y\, w)\,,$$

and the commutation relations (2), (3).

The maps of counit ϵ and antipode S, necessary for a Hopf superalgebra definition, are

$$\begin{aligned} \epsilon(h) &= \epsilon(v_\pm) = 0\,, \quad \epsilon(1) = 1\,, \\ S(h) &= -he^{-\sigma}\,, \quad S(v_-) = -v_- e^{\sigma/2}\,, \quad S(v_+) = -(v_+ - \xi h v_-)e^{-\sigma/2}\,. \end{aligned} \tag{7}$$

We can thus arrive to the following

Definition. The Hopf superalgebra generated by three elements $\{h, v_-, v_+\}$ satisfying the relations (2), (6) and (7) is said to be the twist deformation of $\mathcal{U}(osp(1|2))$ or $osp_\xi(1|2)$.

This is a triangular Hopf superalgebra ($\mathcal{R}_{21}\mathcal{R} = 1$) with universal R-matrix

$$\mathcal{R} = \exp(\frac{1}{2}\,\sigma \otimes h)\,\exp(-\frac{1}{2}\,h \otimes \sigma)\,. \tag{8}$$

The irreducible finite dimensional representations of $osp_\xi(1|2)$

$$\rho_s : osp_\xi(1|2) \longrightarrow End\,(W_s)$$

are the same as for $osp(1|2)$, due to the unchanged algebraic relations (2). They are parametrized by the half-integer spin $s = 0, \frac{1}{2}, 1, \ldots$, have dimension $4s+1$, and are decomposed into a direct sum of two irreps of the $sl(2)$ [13]: $W_s = V_s + V_{s-\frac{1}{2}}$. Hence, the R-matrix in the irreps of $osp_\xi(1|2)$ is a direct sum of four R-matrices of $sl_\xi(2)$. For the first non-trivial case $s = 1/2$ one gets

$$\mathbf{R} = (\rho_{\frac{1}{2}} \otimes \rho_{\frac{1}{2}})\,\mathcal{R} = R(\xi) + I_2 + I_2 + 1\,, \tag{9}$$

where I_2 are 2×2 unit matrices, and $R(\xi)$ is the Jordanian solution to the Yang-Baxter equation (cf. [7])

$$R(\xi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\xi & 1 & 0 & 0 \\ \xi & 0 & 1 & 0 \\ \xi^2 & -\xi & \xi & 1 \end{pmatrix} . \tag{10}$$

The twist parameter can be scaled: $\xi \to \exp(2u)\,\xi$ by the similarity transformation with the element $\exp(-u\,h)$.

The basis of the irreps tensor product decomposition will include deformed Clebsch-Gordan coefficients, expressed as linear combinations of the usual ones and the matrix elements of the twist $\mathcal{F}$ [14]. This is reflected in the spectral decomposition of the R-matrix itself in the tensor product $W_s \otimes W_l$

$$\hat{R}^{s,l} = F^{s,l}\,(\sum_{j=|s-l|}^{s+l} (\pm) P^j)\,(F^{s,l})^{-1} ,$$

where P^j are projectors onto irreducible representations of $osp(1|2)$.

2 Quantum supergroup $OSp_\xi(1|2)$

The self-dual character of the twisted Borel subalgebra $(B_-)_\xi$ was pointed out in [8]. This is obvious in terms of the generators $\{h, \sigma\} \in (B_-)_\xi$ and the generators $\{s, p\} \in (B_-)'_\xi$ of the dual, with the only non-trivial evaluations $\langle h, s\rangle = 2\,,\ \langle \sigma, p\rangle = 2$ [8, 9]:

$$\begin{aligned} [h, \sigma] &= 2\,(1 - e^{\sigma})\,, \qquad [p, s] = 2\,(1 - e^{s})\,, \\ \Delta(\sigma) &= \sigma \otimes 1 + 1 \otimes \sigma\ , \qquad \Delta(s) = s \otimes 1 + 1 \otimes s\ , \\ \Delta(h) &= h \otimes e^{\sigma} + 1 \otimes h\ , \qquad \Delta(p) = p \otimes e^{s} + 1 \otimes p\ , \\ \epsilon(h) &= \epsilon(\sigma) = 0\ , \qquad \epsilon(s) = \epsilon(p) = 0\ , \\ S(h) &= -he^{-\sigma}\ ,\ S(\sigma) = -\sigma\ , \qquad S(p) = -pe^{-s}\ ,\ S(s) = -s\ . \end{aligned}$$

The situation is different for the twisted Hopf super-subalgebra $(sB_-)_\xi$. The latter is generated by two elements $\{h, v_-\}$ as $(B_-)_\xi$. However, due to the Z_2-grading its basis as a linear space consists of even $\sigma^m h^n$ and odd $\sigma^m v_- h^n$ elements ($\sigma = -ln\,(1 + 8\,\xi\,v_-^2)$).

Proposition. The dual $(sB_-)'_\xi$ of the twisted Hopf superalgebra $(sB_-)_\xi$ is generated by three elements $\{\nu, \eta, x\}$ satisfying the relations

$$\begin{aligned} [\nu,\eta] &= 0\,, \quad [\nu,x] = \frac{1}{2}(1-e^{-2\nu})\;, \quad [x,\eta] = \frac{1}{2}\eta\,, \quad \eta^2 = 0\,, \\ \Delta(\nu) &= \nu\otimes 1 + 1\otimes\nu\;, \qquad \Delta(\eta) = \eta\otimes 1 + e^{-\nu}\otimes\eta\;, \\ \Delta(x) &= x\otimes 1 + e^{-2\nu}\otimes x + \frac{1}{8\xi}e^{-\nu}\eta\otimes\eta\;, \\ \epsilon(x) &= \epsilon(\eta) = \epsilon(\nu) = 0\;, \\ S(\eta) &= -\eta e^{\nu}\;, \quad S(\nu) = -\nu\;, \quad S(x) = -xe^{2\nu}\;. \end{aligned} \tag{11}$$

One can check this by a straightforward calculation of evaluating the dual basis $x^k\eta^\delta\nu^l$ of $(sB_-)'_\xi$ and $\sigma^m v_-^\delta h^n$ of $(sB_-)_\xi\,,\ k,l,m,n = 0,1,2,...;\delta = 0,1$ with the only non-zero evaluations among the generators: $\langle h,\nu\rangle = 1\,, \langle v_-,\eta\rangle = 1\ , \langle\sigma,x\rangle = 1$. We shall prove it below by a reduction from the quantum supergroup $OSp_\xi(1|2)$. The universal T-matrix (bicharacter) is given in term of these basis by a product of three exponents

$$\mathcal{T} = \exp(\sigma\otimes x)\,\exp(v_-\otimes\eta)\,\exp(h\otimes\nu)\;.$$

It is interesting to point out that starting from a Hopf superalgebra without nilpotent elements we were forced to introduce Grassmann variables (η) in the dual superalgebra.

The dual of the twisted Hopf superalgebra $osp_\xi(1|2)$ can be introduced using a Z_2-graded version of the FRT-formalism [2], because the R-matrix in the fundamental representation is known (9). The T-matrix of generators of quantum supergroup $OSp_\xi(1|2)$ in this representation has dimension 3×3. There are two convenient basis in this irrep as C^3: i) with grading (0, 1, 0) and ii) with grading (0, 0, 1). The odd generators $v_-,\ v_+$ of $osp(1|2)$ are lower and upper triangular in the former basis, while the latter one is more convenient to write $\mathbf{T}$ in a block matrix form. These forms are

$$\mathbf{T} = \begin{pmatrix} a & \alpha & b \\ \gamma & g & \beta \\ c & \delta & d \end{pmatrix}\;, \qquad \mathbf{T} = \begin{pmatrix} T & \psi \\ \omega & g \end{pmatrix}\;, \tag{12}$$

where T is 2×2 matrix of the even generators $\{a,b,c,d\}$, while ψ and ω are two component column $(\alpha\,,\ \delta)^t$ and row $(\gamma\,,\ \beta)$ vectors of odd elements.

The 3×3 matrix $\mathbf{T}$ of the $OSp_\xi(1|2)$ generators satisfies the FRT-relation

$$\mathbf{R T_1 T_2} = \mathbf{T_2 T_1 R} \tag{13}$$

with $\mathbf{Z}_2$-graded tensor product and 9×9 R-matrix $\mathbf{R}$ (9). From the block-diagonal form of $\mathbf{R}$ (9) it follows for 2×2 matrix T

$$R(\xi)T_1T_2 = T_2T_1R(\xi)\ . \tag{14}$$

Hence, one reproduces the algebraic sector (commutation relations) of the twisted quantum group $SL_\xi(2)$ for the generators $\{a, b, c, d\}$ [7]. For the other blocks of different dimension we get from (13)

$$\begin{aligned} R(\xi)T_1\psi_2 &= \psi_2 T_1\ , \quad g\,\mathbf{T} = \mathbf{T}\,g\ , &(15)\\ \omega_1 T_2 &= T_2\omega_1 R(\xi)\ , \quad \omega_1\psi_2 = -\psi_2\omega_1\ , &(16)\\ \omega_1\omega_2 &= -\omega_2\omega_1 R(\xi)\ ,\ R(\xi)\psi_1\psi_2 = -\psi_2\psi_1\ . &(17)\end{aligned}$$

From the relations (14) - (17) one gets centrality of the following elements:

$$\det{}_\xi T = a(d-\xi b) - cb\ , \quad g\ , \quad \theta = \omega\, T^{-1}\,\psi\ .$$

Coproduct, counit and antipode are given by the standard expressions of the FRT-formalism [2]

$$\Delta(\mathbf{T}) = \mathbf{T}\otimes\mathbf{T}\ , \quad \epsilon(\mathbf{T}) = I_3\ , \qquad S(\mathbf{T}) = \mathbf{T}^{-1}\ . \tag{18}$$

The inverse of $\mathbf{T}$ is expressed in terms of the generators (12) provided invertability of $\det_\xi T$, and $(g-\omega\, T^{-1}\,\psi)$

$$\mathbf{T}^{-1} = \begin{pmatrix} I_2 & -T^{-1}\psi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} T^{-1} & 0 \\ 0 & (g-\theta)^{-1}\end{pmatrix}\begin{pmatrix} I_2 & 0 \\ -\omega T^{-1} & 1\end{pmatrix}\ . \tag{19}$$

Thus we arrive to the following
Definition. The dual to the Hopf superalgebra $osp_\xi(1|2)$ generated by the entries of $\mathbf{T}$ (12) subject to the relations (14) - (18) is said to be the quantum supergroup $OSp_\xi(1|2)$.

Another way to define this $OSp_\xi(1|2)$ is to use the twist element $\mathcal{F}$ as the pseudodifferential operator on the Lie supergroup $OSp(1|2)$, and redefine super-commutative product of functions on this supergroup.

The reduction or Hopf superalgebra homomorphism, of $OSp_\xi(1|2)$ to $(sB_-)'_\xi$ is given by :

$$b = \alpha = \beta = 0\,,\ g = 1\,,\ a = d^{-1} = \exp(\nu)\,,\ \gamma\, a^{-1} = \delta = \frac{1}{2}\eta\,,\ c = 2\xi x a\,.$$

3 Conclusion

Using embedding of the Lie algebra $sl(2)$ into the rank one orthosymplectic superalgebra the latter one was deformed by the twist element $\mathcal{F} \in \mathcal{U}(sl(2))^{\otimes 2}$. Although the deformed Lie superalgebra is finite dimensional it can be used for further deformation of infinite dimensional Hopf superalgebras (e.g. super-Yangians) and integrable models [14]. There are also possibilities for different contractions. The work in this direction is in progress.

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